Lecture 5 **2012/10/29**

Outline

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1. Motivation

- Existence of solutions on finite time intervals is very restrictive and in most cases we need existence on $[t_0, \infty)$ for any $(t_0, x_0) \in R \times R^n$.
- Comparison principle is an important tool to estimate bounds on the solutions without solving ODE.
- Comparison principle is variation of inequality techniques.

2. Global Existence

1) Linear Boundedness

Definition 5.1 $f : R \times R^n \to R^n$ is **linearly bounded** if there exist $a \geq 0$ and $b \geq 0$ such that

$$\| f(t,x) \| \leq a \| x \| + b \quad \text{for all} \quad (t,x) \in R \times R^n .$$

Theorem 5.1 Suppose that $f(t,x)$ is continuous; locally Lipschitz on $R \times R^n$ and linearly bounded. Then the unique solution $x(t)$ of (E) has $I^+_{\max} = [t_0, \infty)$ for any $x_0 \in R^n$.

Proof. Suppose that $I^+_{\max} = [t_0, \omega_+)$. We show $\omega_+ = \infty$. Since

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) ds ,$$

we have

$$\| x(t) \| \leq \| x_0 \| + \int_{t_0}^{t} \| f(s, x(s) \| ds \leq \| x_0 \| + \int_{t_0}^{t} (a \| x(s) \| + b) ds$$

$$\leq (\| x_0 \| + b(t - t_0)) + a \int_{t_0}^{t} \| x(s) \| ds .$$

Application of Gronwall's inequality yields

$$\| x(t) \| \le (\| x_0 \| + b(t - t_0)) e^{a(t-t_0)} .$$

If $\omega_+ < \infty$, then

$$\| x(t) \| \le (\| x_0 \| + b(\omega_+ - t_0)) e^{a(\omega_+ - t_0)} < \infty .$$

This is not possible by the continuation theorem. This shows that $\omega_+ = \infty$. □

Remark 5.1 Even for a linear system $x' = A(t)x + h(t)$, where $A(t)$, $h(t) \in C(R)$, not necessarily bounded on $(-\infty, \infty)$, so it is not necessary to satisfy the linear boundedness. Although the linearly bounded is rather restrictive, it is easy to be checked.

2) Additional Lyapunov Condition

Theorem 5.2 Let $f(t,x)$ be continuous and locally Lipschitz on $R \times R^n$. Suppose that there exist $V(t,x): R \times R^n \to R$ of class C^1 such that

- $W_1(x) \le V(t,x) \le W_2(x)$ where $W_1(x) \ge 0$ with $W_1(x) = 0 \Rightarrow x = 0$;

  (**positively definite**)
- $\lim_{\|x\| \to \infty} W_1(x) = \infty$, (**radially unbounded)**;
- $\dot{V}(t,x) \overset{def\cdot}{=} \dfrac{\partial V}{\partial t} + \dfrac{\partial V}{\partial x} \cdot f(t,x) \le a + bW_1(x)$.

Then the unique solution $x(t)$ of (E) has $I^+_{\max} = [t_0, \infty)$ for any $x_0 \in R^n$.

Proof. By the above Lyapunov conditions, we have

$$\frac{d}{dt} V(t, x(t)) = \frac{\partial V(t, x(t))}{\partial t} + \frac{\partial V(t, x(t))}{\partial x} \cdot f(t, x(t)) \le a + bW_1(x(t))$$

$$\le a + bV(t, x(t)) .$$

Integrating this inequality yields

$$V(t, x(t)) \le V(t_0, x_0) + \int_{t_0}^{t} (a + bV(s, x(s)) ds .$$

Application of Gronwall's inequality gives the bound

$$V(t, x(t)) \le \{ a(t - t_0) + V(t_0, x_0) \} e^{b(t-t_0)} .$$

That is,

$$W_1(x(t)) \le \{a(t-t_0)+V(t_0,x_0)\}e^{b(t-t_0)}.$$

If $I_{\max}^{+}=[t_0,\omega_+)$ with $\omega_+<\infty$, then

$$W_1(x(t)) \le \{a(\omega_+-t_0)+V(t_0,x_0)\}e^{b(\omega_+-t_0)}<\infty. \qquad \text{(F1)}$$

By the continuation theorem, it must have $\lim_{t\to\omega_+}\|x(t)\|=\infty$. Then,

$$\lim_{t\to\omega_+}W_1(x(t))=\lim_{\|x\|\to\infty}W_1(x)=\infty.$$

This contradicts (F1). This contradiction shows that $\omega_+=\infty$. □

Corollary 5.1 Let $f(x)$ be locally Lipschitz on R^n. Suppose that there exist $V(x):R^n\to R$ of class C^1 such that

- $V(x)\ge 0$ with $V(x)=0 \;\Rightarrow\; x=0$; (**positively definite**)
- $\lim_{\|x\|\to\infty}V(x)=\infty$, (**radially unbounded)**;
- $\dot{V}(x)\overset{def.}{=}\frac{\partial V}{\partial x}\cdot f(t,x)\le a+bV(x)$.

Then the unique solution $x(t)$ of (E) has $I_{\max}^{+}=[0,\infty)$ for any $x_0\in R^n$.

Proof. Since $f(x)$ is free of t, then we take $V(x)=V(t,x)\equiv W_1(x)\equiv W_2(x)$ and $t_0=0$. □

Remark 5.2 How to find a desired Lyapunov candidate, there is no systematic way in general to get it. It is still an open problem in Math. However, existence of Lyapunov function is guarantied under reasonable mild conditions. The details will be given in Lyapunov stability theory.

3. Some Important Class of Systems with Global Existence

1) Gradient Systems

Suppose that $V(x):R^n\to R_{\ge 0}$ is a function of C^2.

$$x'=-\nabla V(x),$$

is called a gradient system, where $\nabla V(x) = \left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \cdots, \frac{\partial V}{\partial x_n} \right)^T = \left(\frac{\partial V}{\partial x} \right)^T$.

Lemma 5.1 Suppose that $V(x): R^n \to R_{\geq 0}$ is a function of C^2 with $\lim_{\|x\| \to \infty} V(x) = \infty$. Then any solution of the gradient system exists for all $t \geq 0$.

Proof. Making derivative along trajectories of the gradient system, we have

$$\frac{d}{dt} V(x(t)) = \frac{\partial V}{\partial x} \cdot x' = -\frac{\partial V}{\partial x} \cdot \left(\frac{\partial V}{\partial x} \right)^T = -\| \frac{\partial V(x)}{\partial x} \|^2 \leq 0.$$

We know that $V(x(t)) \leq V(x_0)$ for all $t \geq 0$. From which we conclude that $I_{\max} = [0, +\infty)$. Otherwise, there exists a time $\omega_+ < \infty$ s.t. $\overline{\lim_{t \to \omega_+^-}} \| x(t) \| = \infty$ by the continuation theorem. Then, there exists $\{t_n\} \to \omega_+^-$ as $n \to \infty$ s.t. $\lim_{\|x(t_n)\| \to \omega_+^-} V(x(t_n)) = \infty$. This contradicts $V(x(t)) \leq V(x_0)$. So $I_{\max} = [0, +\infty)$. □

Remark 5.3 If $x \in V^{-1}(c) = \{x : V(x) = c\}$ is a regular point (i.e. $\nabla V(x) \neq 0$), the solution curve $x(t)$ is perpendicular to the level surface $V^{-1}(c)$. Since for any curve $\gamma(t) \in V^{-1}(c)$ with $\gamma(0) = x$ and $\gamma'(0) = y$, we have

$$0 = \frac{d}{dt} V(\gamma(t)) |_{t=0} = \frac{\partial V(\gamma(t))}{\partial x} \cdot \gamma'(t) |_{t=0} = \nabla V(x))^T \cdot y = \langle \nabla V(x), \quad y \rangle.$$

2) Hamiltonian Systems

Suppose that $H(x, y): R^n \times R^n \to R_{\geq 0}$ is a function of C^2.

$$x' = \nabla_y H(x, y); \quad y' = -\nabla_x H(x, y)$$

is called the Hamiltonian equation, where $H(x, y)$ is called a Hamiltonian function.

Since $f(x) = \left(\nabla_y H(x, y), \quad -\nabla_x H(x, y) \right)^T$ is locally Lipschitz, so the existence and uniqueness of solution is done. Suppose that $(x(t), y(t))$ is a solution. Then,

$$\frac{d}{dt} H(x(t), y(t)) = \nabla_x H(x(t), y(t)) x'(t) + \nabla_y H(x(t), y(t)) y'(t) \equiv 0.$$

$$\Rightarrow \quad H(x(t), y(t)) \equiv \text{const.}$$

$H(x, y)$ can be regarded as a Lyapunov candidate for the Hamiltonian equation. If $\lim_{\|(x,y)\|\to\infty} H(x, y) = \infty$, then the level set $\{(x, y): H(x, y) = c\}$ is closed and bounded. We conclude that $I_{\max} = (-\infty, +\infty)$. Otherwise, there exists a time $\omega_+ < \infty$ ($\omega_- > -\infty$) s.t. $\overline{\lim}_{t\to\omega_+^-(\omega_-^+)} \|(x(t), y(t))\| = \infty$. From which it yields that there exist

$$(x(t_n), y(t_n)) = (x_n, y_n) \in \{(x, y): H(x, y) = c\} \ \text{ s.t. } \lim_{\|(x_n, y_n)\|\to\infty} H(x_n, y_n) = \infty.$$

However, this is not possible. □

3) Van der Pol Equation

$$x'' = \varepsilon(1 - x^2)x' - x$$

is called Van der Pol equation, where $\varepsilon > 0$ is a small parameter. The form of system:

$$\begin{cases} x' = y \\ y' = \varepsilon(1 - x^2)y - x \end{cases},$$

can be regarded as a perturbation of a particular Hamiltonian system:

$$\begin{cases} x' = y \\ y' = -x \end{cases}.$$

From which we find $H(x, y) = \frac{1}{2}(x^2 + y^2)$ satisfying $\lim_{\|x\|\to\infty} H(x, y) = \infty$, which can be taken as a Lyapunov candidate for the Van der Pol equation. Then we have

$$\frac{d}{dt}H(x, y) = \frac{\partial H}{\partial x}x' + \frac{\partial H}{\partial y}y' = \varepsilon(1 - x^2)y^2 \le \begin{cases} 0, & x^2 \ge 1 \\ \varepsilon y^2, & x^2 \le 1 \end{cases} \le 2\varepsilon H(x, y).$$

By Corollary 5.1, we obtain the global existence.

4) Dissipative Systems

Let $f: R^n \to R^n$ be locally Lipschitz. Suppose that there exist $v \in R^n$, and $a > 0$, $b > 0$ s.t.

$$\langle f(x), \quad x - v \rangle \le a - b\|x\|^2.$$

Then the IVP $x' = f(x)$, $x(0) = x_0$, has a unique solution $x(t)$ for $t \ge 0$.

Proof. Taking a ball $B_0 = \{x \in R^n : \| x \|^2 \le \frac{a}{b}\}$ and a Lyapunov candidate as follows.

$$V(x) = \frac{1}{2} \| x - v \|^2$$

satisfying $\lim_{\|x\| \to \infty} V(x) = \infty$, we have

$$\frac{d}{dt} V(x(t)) = \langle f(x), \quad x - v \rangle \le a - b \| x \|^2 < 0 \quad \text{as} \quad \| x \|^2 > \frac{a}{b}.$$

This implies the global existence by Corollary 5.1.

Remark 5.4 The general definition of dissipative systems for $x' = f(x)$ is given as follows.

If there exists a bound $B > 0$ s.t. for any solution $x(t)$ of $x' = f(x)$, $x(0) = x_0$, there exists a sufficiently large constant $T(x_0) > 0$, s.t

$$t \ge T(x_0) \quad \Rightarrow \quad \| x(t) \| < B,$$

then $x' = f(x)$ is called a dissipative system. Obviously, the above system is dissipative.

5) Lorentz Equations

The Lorentz equations are given by

$$\begin{cases} x_1' = -\sigma x_1 + \sigma x_2 \\ x_2' = -x_1 x_3 + r x_1 - x_2 \\ x_3' = x_1 x_2 - b x_3 \end{cases},$$

where $\sigma > 0$, $r > 0$ and $b > 1$ are system parameters. (Note: when $r > r_0 = 24.74$, it could exhibit chaotic behavior)

Taking $v = (0, \quad 0, \quad \gamma)$, where $\gamma = \sigma + r$, we have

$$\begin{aligned} \langle f(x), x - v \rangle &= -\sigma x_1^2 - x_2^2 - b x_3^2 + (\sigma + r - \gamma) x_1 x_2 + b \gamma x_3 \\ &= -\sigma x_1^2 - x_2^2 - b x_3^2 + b \gamma x_3 \\ &\le -\sigma x_1^2 - x_2^2 - \frac{b}{2} x_3^2 + b \frac{\gamma^2}{2} \\ &= a - b(x_1^2 + x_2^2 + x_3^2) = a - b \| x \|^2, \end{aligned}$$

where $a = b\frac{\gamma^2}{2}$ and $b = \min\{\sigma, 1, \frac{b}{2}\}$. So the Lorentz equations are dissipative.

4. Comparison Principle

1) Dini Derivative

$$D^+v(t) = \overline{\lim_{h\to 0^+}}\frac{v(t+h)-v(t)}{h},$$

where $v: R \to R$. When the right limit is unique, we have a right hand derivative as follows.

$$D_r v(t) = \lim_{h\to 0^+}\frac{v(t+h)-v(t)}{h}.$$

2) Comparison Lemma

Lemma 5.2 (Comparison Lemma) Consider the scalar function $f(t,u)$ is continuous and locally Lipschitz, where $t \ge t_0$ and $u \in R$. If

$$u'(t) = f(t, u(t)),\ u(t_0) = u_0;$$

$$D_r v(t) \le f(t, v(t)),\ v(t_0) = v_0$$

with $v_0 \le u_0$, then $v(t) \le u(t)$ on any compact interval $t \in [t_0, t_1]$.

Proof. Consider

$$z'(t) = f(t, z(t)) + \frac{1}{n} \overset{\Delta}{=} F(t, z(t), n),\ z(t_0) = u_0, \tag{C.1}$$

where $n \in N^+$. On any $[t_0, t_1]$, we conclude from the continuous dependence that for any $\varepsilon > 0$, there exists $n_0 \ge 1$ s.t. $n \ge n_0$, the IVP (C.1) has a unique solution $z(t,n)$, defined on $[t_0, t_1]$ and

$$|z(t,n) - u(t)| \le \varepsilon,\ \ t \in [t_0, t_1]. \tag{C.2}$$

Claim 1: $v(t) \le z(t,n)$**,** $t \in [t_0, t_1]$, $n \ge n_0$.

Show by contradiction. If it were not the case, there would be times $a, b \in [t_0, t_1]$ s.t. $v(a) = z(a,n)$ and $v(t) > z(t,n)$ for all $a < t \le b$. Consequently,

$$v(t) - v(a) > z(t,n) - z(a,n),\ \ t \in (a, b],$$

$$\Rightarrow \qquad \frac{v(t)-v(a)}{h} > \frac{z(t,n)-z(a,n)}{h}, \quad t \in (a,b],$$

$$\Rightarrow \qquad D_r v(a) \geq z'(a,n) = f(a, z(a,n)) + \frac{1}{n} > f(a, z(a,n)) = f(a, v(a)),$$

which contradicts the condition $D_r v(t) \leq f(t, v(t))$ for all $t \in [t_0, t_1]$.

Claim 2: $v(t) \leq u(t)$, $t \in [t_0, t_1]$, $n \geq n_0$.

Show by contradiction. If it were not the case, there would be time $a \in (t_0, t_1]$ s.t. $v(a) > u(a)$. Taking $\varepsilon = \frac{v(a)-u(a)}{2} > 0$ and using (C.2), we have

$$v(a) - z(a,n) = v(a) - u(a) + u(a) - z(a,n) \geq 2\varepsilon - \varepsilon = \varepsilon,$$

which contradicts Claim 1. □

Example 5.1 Find the bound of solution for the IVP $x' = f(x) = -(1+x^2)x$, $x(0) = a$ without solving the equation.

Solution. It has a unique solution on $[0, \omega_+)$ for some certain $\omega_+ > 0$ (ω_+ could be infinity) because $f(x)$ is locally Lipschitz continuous. Let $v(t) = x^2(t)$. Then $v(t) \in C^1$ and $v'(t) = 2x(t)x^2(t) = -2x^2(t) - 2x^4(t) \leq -2x^2(t)$. Hence,

$$v'(t) \leq -2v(t), \quad v(0) = a^2.$$

Consider the IVP $u' = -2u$, $u(0) = a^2 \Rightarrow u(t) = a^2 e^{-2t}$. Then, by the comparison lemma, the solution $x(t)$ is defined on any compact interval $[0, t_1] \subset [0, \omega_+)$, and satisfies

$$|x(t)| = \sqrt{v(t)} \leq e^{-t} |a|, \quad t \in [0, t_1].$$

First, we say that the above inequality holds for $[0, \omega_+)$ by continuation theorem. Then we conclude that the inequality holds for all $t \geq 0$. If it were not the case, it would be time $\omega_+ < \infty$ s.t. $\overline{\lim_{t \to \omega_+^-}} |x(t)| = \infty$ by the continuation theorem. However, this is not possible because $|x(t)| \leq e^{-\beta} |a| < \infty$ for all $t \geq 0$. Therefore,

$$|x(t)| = \sqrt{v(t)} \leq e^{-t} |a|, \quad \forall t \geq 0. \;\; \square$$

Example 5.2 Find the bound of solution for the IVP $x' = f(t,x) = -(1+x^2)x + e^t$, $x(0) = a$ without solving the equation. (**Homework**)

3) An Important Lemma for a Vector Function

Lemma 5.3 Suppose that $x(t) \in C^1([a,b])$ is an n-vector valued function, then $D_r(\| x(t) \|)$ exists on $a \le t < b$ and $D_r(\| x(t) \|) \le \| x'(t) \|$, $a \le t < b$.

Proof. For the existence of $D_r(\| x(t) \|)$, For any $x, y \in R^n$ and $0 < \theta \le 1$, $h > 0$, we have (Δ inequality)

$$\| x + \theta h y \| - \| \theta x + \theta h y \| \le \| x - \theta x \| \le (1-\theta) \| x \|,$$

$$\Rightarrow \qquad \| x + \theta h y \| - \| x \| \le \theta (\| x + h y \| - \| x \|);$$

$$\Rightarrow \qquad \frac{\| x + \theta h y \| - \| x \|}{\theta h} \le \frac{\| x + h y \| - \| x \|}{h}.$$

This implies that $\frac{\| x + h y \| - \| x \|}{h}$ is non-decreasing on $h > 0$. Moreover, it is bounded below by $-\| y \|$ since $\| x + h y \| \ge \| x \| - h \| y \|$. Therefore

$$\lim_{h \to 0^+} \frac{\| x + h y \| - \| x \|}{h} \text{ exists.}$$

Since $x(t) \in C^1([a,b])$**,** then the latter limit implies

$$\lim_{h \to 0^+} \frac{\| x(t) + h x'(t) \| - \| x(t) \|}{h} \text{ exists.}$$

Since

$$| \{ \| x(t+h) \| - \| x(t) \| \} - \{ \| x(t) + h x'(t) \| - \| x(t) \| \} |$$

$$= | \{ \| x(t+h) \| - \| x(t) + h x'(t) \| \} |$$

$$\le | \{ \| x(t+h) - x(t) - h x'(t) \| \} | = o(h), \text{ (Tailor Expansion)}$$

as $h \to 0^+$, it follows that

$$\lim_{h \to 0^+} \frac{\| x(t+h) \| - \| x(t) \|}{h} = \lim_{h \to 0^+} \frac{\| x(t) + h x'(t) \| - \| x(t) \|}{h} \text{ exists.}$$

Therefore,

$$D_r(\|x(t)\|)=\lim_{h\to 0^+}\frac{\|x(t+h)\|-\|x(t)\|}{h} \text{ exists.}$$

Since

$$\frac{\|x(t+h)\|-\|x(t)\|}{h} \le \frac{\|x(t+h)-x(t)\|}{h} \quad \text{for } h>0,$$

take the limit as $h\to 0^+$ on both sides to obtain $D_r(\|x(t)\|) \le \|x'(t)\|$ and the proof is completed. □

Remark 5.5 This lemma shows that once $D_r(\|x(t)\|)$ exists, derivative sign and norm sign can exchange with the inequality relation $D_r(\|x(t)\|) \le \|x'(t)\|$. It looks seemly nothing to do with ODE. However, it is important for ODE with the bound estimation of solution.

4) Comparison Theorem for the Global

Theorem 5.3 (**Comparison Theorem**) Suppose that $f(t,x)$ of (E) is continuous and locally Lipschitz, where $t\in[t_0,\omega^+)$ (ω^+ could be infinity) and $x\in R^n$; and satisfies

$$\|f(t,x)\|\le F(t,\|x\|),\ (t,x)\in[t_0,\omega^+)\times R^n,$$

and $\|x(t_0)\|\le\eta$, where the IVP of the scalar equation

$$u'=F(t,u),\ u(t_0)=\eta$$

has a unique solution $u(t)$ for $t\in[t_0,\omega^+)$. Then, $x(t)$ exists on $t\in[t_0,\omega^+)$ and

$$\|x(t)\|\le u(t) \quad \text{for all } t\in[t_0,\omega^+).$$

Proof. Let $v(t)=\|x(t)\|$. Then

$$D_r v(t)=D_r\|x(t)\| \le \|x'(t)\|=\|f(t,x(t))\| \le F(t,\|x(t)\|)=F(t,v(t))$$

and $v(t_0)=\|x(t_0)\|\le\eta$. Application of Lemma 5.1 (**the comparison lemma**) yields

$$\|x(t)\|\le u(t)$$

for any compact interval of $[t_0,\omega^+)$. We conclude that $\|x(t)\|\le u(t)$ for all $t\in[t_0,\omega^+)$. Show by contradiction. If it were not the case, it would be a time c with

$t_0 < c < \omega^+$ s.t. $\overline{\lim_{t \to c^-}} \| x(t) \| = \infty$ by the continuation theorem. But this is not possible because $\| x(c) \| \le u(c) < \infty$. □

Remark 5.6 The result of Theorem 5.3 (**Comparison Theorem**) is global!! It doesn't matter if Lipschitz condition is not satisfied. However, the uniqueness of solution is not guaranteered.

Remark 5.7 Finding $u(t)$ is a key in application of this comparison theorem. For example, $F(t,u) = au + b$ (**Linear Equation**); $F(t,u) = au + bu^n$ (**Bernoulli Equation**); $F(t,u) = g(t)F(u)$ (**Wintner Theorem**) and the others (**DIY**), $u(t)$ can be found.

5) Some Important Applications

Theorem 5.4 (Wintner Theorem) Suppose that in Theorem 5.3, if

$$\| f(t,x) \| \le g(t)\, L(\| x \|),$$

where $g(t) \ge 0$ is continuous for $t \ge t_0$ and $L(u) \ge 0$ is continuous for $u > 0$, and satisfies

$$\int_{u_0}^{+\infty} \frac{du}{L(u)} = +\infty,$$

then the solution $u(t)$ of $u' = g(t)\, F(u)$, $u(t_0) = u_0 > 0$, with $\| x(t_0) \| = u_0$ exists for all $t \ge t_0$ and satisfies

$$\| x(t) \| \le u(t)$$

for all $t \ge t_0$.

Proof. By the comparison theorem, we only need to show the existence of $u(t; t_0, u_0)$ for all $t \ge t_0$. Since $u(t)$ satisfies

$$\int_{u_0}^{u} \frac{du}{L(u)} = \int_{t_0}^{t} g(s) ds,$$

if $u(t)$ would not exist globally on $t \ge t_0$, there would be a finite escape. Then there exists $\omega_+ < \infty$ and $\{t_n\}$ s.t. $\lim\limits_{t_n \to \omega_+^-} u(t_n) = \infty$. That is,

$$\int_{u_0}^{u(t_n)} \frac{du}{L(u)} = \int_{t_0}^{t_n} g(s)ds .$$

But, this is not possible because the left is ∞ and the right is finite. □

Theorem 5.5 For linear equations $x' = A(t)x + h(t)$, where $A(t), h(t) \in C(R)$, then $I_{\max} = [t_0, +\infty)$, $t_0 \in R$.

Proof. In fact,

$$\| A(t)x + h(t) \| \le \| A(t) \| \| x \| + \| h(t) \|$$

$$\le \max\{\| A(t) \|, \| h(t) \|\}(\| x \| + 1) = g(t)\, L(\| x \|) .$$

Since $L(u) = u + 1$ is continuous and locally Lipschitz, and $\int_0^u \frac{du}{u+1} = \infty$, we have the desired result by Wintner theorem. □

Remark 5.8 You may prove Theorem 5.5 with $I_{\max} = (-\infty, +\infty)$ by Gronwall's inequality. (**Homework**)

5. Summary

- **We introduced three main methods for the global existence: the linear boundedness, Lyaponove method and the comparison method.**

- **Several important classes of systems have the global existence.**